Math 150, *Lecture Notes- Bonds* **Name_____________**

Mean

Section 3.2 *Rolle's Theorem and the Man Value Theorem*

Rolle's Theorem

The Extreme Value Theorem (Section 3.1) states that a continuous function on a closed interval $[a, b]$ must have both a minimum and a maximum on the interval. Both of these values, however, can occur at the endpoints. **Rolle's Theorem,** named after the French mathematician Michel Rolle (1652–1719), gives conditions that guarantee the existence of an extreme value in the *interior* of a closed interval.

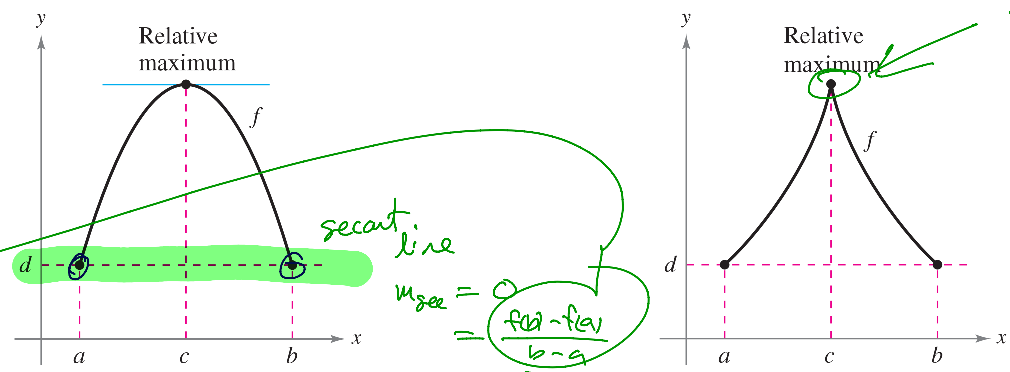
THEOREM 3.3 Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If

$$f(a) = f(b)$$

then there is at least one number c in (a, b) such that $f'(c) = 0$.

f'(x) does not exist at x=c



(a) f is continuous on $[a, b]$ and differentiable on (a, b).

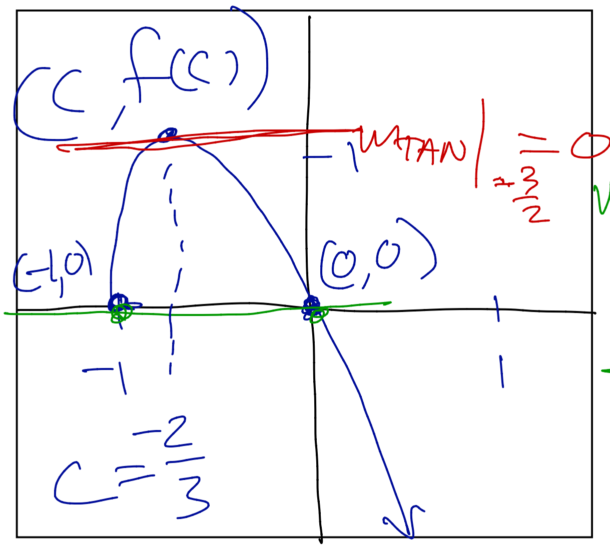
(b) f is continuous on $[a, b]$.

Figure 3.8

From Rolle's Theorem, you can see that if a function f is continuous on $[a, b]$ and differentiable on (a, b), and if $f(a) = f(b)$, there must be at least one x-value between a and b at which the graph of f has a horizontal tangent, as shown in Figure 3.8(a). If the differentiability requirement is dropped from Rolle's Theorem, f will still have a critical number in (a, b), but it may not yield a horizontal tangent. Such a case is shown in Figure 3.8(b).

$[-1, 0] = [a, b]$

Ex.1 Find the two x-intercepts of $f(x) = -3x\sqrt{x+1}$ and show that $f'(x) = 0$ at some point between the two intercepts.



$m_{secant}\Big|_{[-1,0]} = 0$

$$m_{sec} = \frac{f(0) - f(-1)}{(-1) - (0)} = \frac{0 - 0}{-1} = 0$$

$$f'(x) = \frac{d}{dx}\left[-3x\,(x+1)^{1/2}\right]$$

$$f'(x) = (-3x)\frac{d}{dx}\left[(x+1)^{1/2}\right] + (x+1)^{1/2}\cdot\frac{d}{dx}(-3x)$$

$$f'(x) = -3x\left[\tfrac{1}{2}(x+1)^{-1/2}\cdot 1\right] + (x+1)^{1/2}(-3)$$

chain Rule

$$f'(x) = -\frac{3x}{2}(x+1)^{-1/2} - 3(x+1)^{1/2}$$

$$f'(x) = -3(x+1)^{-1/2}\left[\frac{x}{2} + (x+1)^{2/2}\right]$$

$$f'(x) = -3(x+1)^{-1/2}\left[\frac{x}{2} + \frac{2x}{2} + 1\right]$$

$$f'(x) = -3(x+1)^{-1/2}\left[\frac{3x}{2} + 1\right]$$

solve:

$$f'(x) = 0$$

$$-3(x+1)^{-1/2}\left(\frac{3x}{2} + 1\right) = 0$$

$$\frac{-3\left(\frac{3x}{2}+1\right)}{\sqrt{x+1}} = 0$$

$$-3\cdot\left(\tfrac{3}{2}x + 1\right) = 0$$

$$\frac{-3\cdot\left(\frac{3x}{2}+1\right)}{-3} = \frac{0}{-3}$$

$$\tfrac{3}{2}x + 1 = 0$$

$$\tfrac{2}{3}\cdot\tfrac{3}{2}x = -1\cdot\tfrac{2}{3}$$

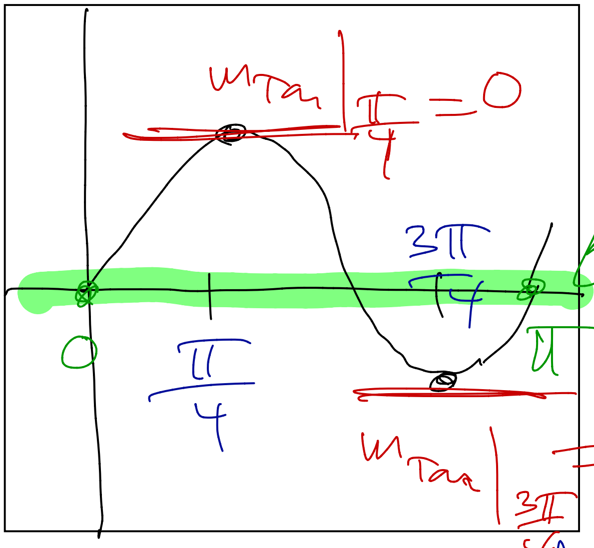
$$x = \frac{-2}{3} = c$$

$$-1 \le \frac{-2}{3} \le 0$$

$$f'(c) = 0$$

$$f'\left(\frac{-2}{3}\right) = 0$$

Ex.2 Let $g(x)=\sin(2x)$. Find all values of c in the interval $(0,\pi)$ such that $g'(c)=0$.



Secant line through $(0,0)$ & $(\pi,0)$

$m_{sec} = 0$

$$g'(x) = \frac{d}{dx}\left[\sin(2x)\right]$$

$$g'(x) = \underbrace{\cos(2x)\cdot 2}_{\text{chain rule}}$$

$$g'(x) = 2\cos(2x)$$

$$g'(x) = 0$$

$$2\cos(2x) = 0$$

$$\cos(2x) = 0$$

$$2x = \frac{\pi}{2} \quad \& \quad \frac{3\pi}{2} = 2x$$

$$x = \frac{\pi}{4} \qquad \frac{3\pi}{4} = x$$

THEOREM 3.4 The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

NOTE The "mean" in the Mean Value Theorem refers to the mean (or average) rate of change of f in the interval $[a, b]$. ■

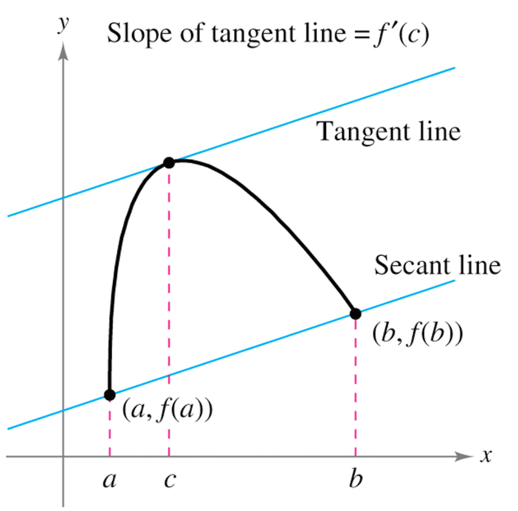


Figure 3.12

PROOF Refer to Figure 3.12. The equation of the secant line that passes through the points $(a, f(a))$ and $(b, f(b))$ is

$$y = \left[\frac{f(b) - f(a)}{b - a}\right](x - a) + f(a).$$

Let $g(x)$ be the difference between $f(x)$ and y. Then

$$\begin{aligned} g(x) &= f(x) - y \\ &= f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a) - f(a). \end{aligned}$$

By evaluating g at a and b, you can see that $g(a) = 0 = g(b)$. Because f is continuous on $[a, b]$, it follows that g is also continuous on $[a, b]$. Furthermore, because f is differentiable, g is also differentiable, and you can apply Rolle's Theorem to the function g. So, there exists a number c in (a, b) such that $g'(c) = 0$, which implies that

$$\begin{aligned} 0 &= g'(c) \\ &= f'(c) - \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

So, there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$ ■

On a Test

The Mean Value Theorem has implications for both basic interpretations of the derivative. Geometrically, the theorem guarantees the existence of a tangent line that is parallel to the secant line through the points $(a, f(a))$ and $(b, f(b))$, as shown in Figure 3.12. Example 3 illustrates this geometric interpretation of the Mean Value Theorem. In terms of rates of change, the Mean Value Theorem implies that there must be a point in the open interval (a, b) at which the instantaneous rate of change is equal to the average rate of change over the interval $[a, b]$. This is illustrated in Example 4.

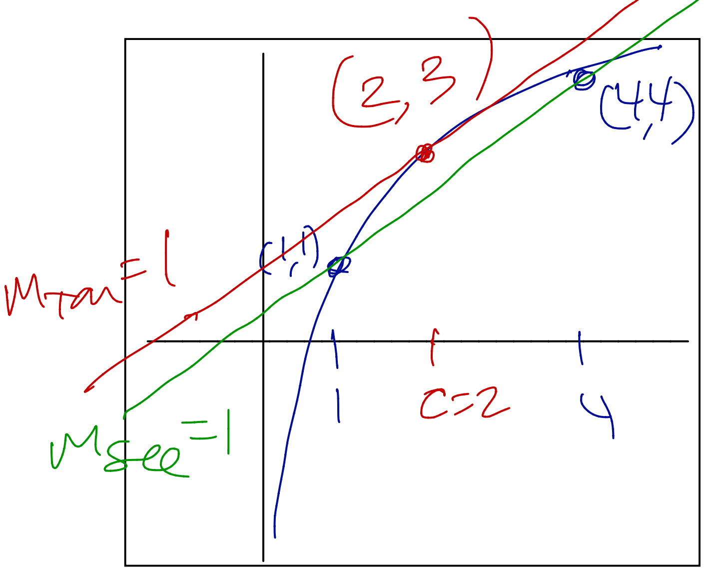
$h(4) = 5 - \frac{4}{4}$
$= 5 - 1$
$h(4) = 4$

$h(1) = 5 - \frac{4}{1} = 5 - 4 = 1$

Ex.3 Let $h(t) = 5 - \frac{4}{t}$. Find all values of c in the interval $(1,4)$ such that $h'(c) = \frac{h(4) - h(1)}{4 - 1}$.

$h(t) = 5 - 4t^{-1}$
$h'(t) = 0 - 4[-1 \cdot t^{-2}]$
$h'(t) = 4t^{-2}$

$h'(t) = \frac{4}{t^2}$



$h'(c) = \frac{h(4) - h(1)}{4 - 1} = m_{sec\,[1,4]}$

$= \frac{4 - 1}{3}$

$= \frac{3}{3}$

$\boxed{h'(c) = 1} = m_{sec\,[1,4]}$

M.V.T.

$h'(c) = 1$

$\frac{4}{c^2} = 1$

$\frac{4}{1} = c^2$

$4 = c^2$

$c = 2$ or $c = -2$

No ∇ Not in [1,4]

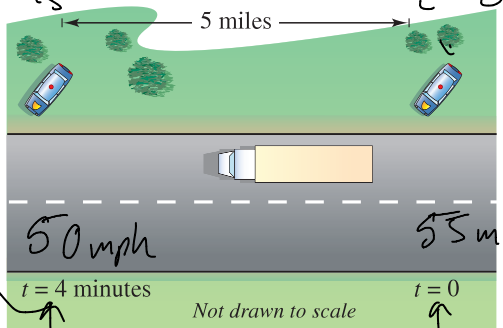
$h(2) = 5 - \frac{4}{2}$
$h(2) = 5 - 2$
$h(2) = 3$

Ex.4 Finding an Instantaneous Rate of Change

Two stationary patrol cars equipped with radar are 5 miles apart on a highway, as shown in Figure 3.14. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit (of 55 miles per hour) at some time during the 4 minutes.

$s(\frac{1}{15}) = 5$ miles

$t = \frac{1}{15}$ hour $\qquad t = 0$ hours



$s(0) = 0$ miles

At some time t, the instantaneous velocity is equal to the average velocity over 4 minutes.

Figure 3.14

Let $s(t)$ = the distance traveled by the truck (miles)

$s'(t) = V(t)$ = velocity of the truck (mph)

$t = 4$ minutes $= \frac{4 \text{ minutes}}{1}\left(\frac{1 \text{ hour}}{60 \text{ minutes}}\right)$

$t = \frac{1}{15}$ hour

$$\text{Average velocity} = \frac{s(b) - s(a)}{b - a} = \frac{\text{miles}}{\text{hour}}$$

$$= \frac{s(\frac{1}{15}) - s(0)}{\frac{1}{15} - 0}$$

$$= \frac{5 \text{ miles} - 0 \text{ miles}}{\frac{1}{15} \text{ hour}}$$

$$= 5 \cdot \frac{15}{1} \frac{\text{miles}}{\text{hour}}$$

$$\text{Average Velocity} = 75 \frac{\text{miles}}{\text{hour}}$$

If $s(t)$ is differentiable, the mean value theorem says that there is a c in $(0, \frac{1}{15})$ such that

$$v(c) = s'(c) = \frac{s(b) - s(a)}{b - a} \quad \text{on } [a,b] = [0, \tfrac{1}{15}]$$

$$v(c) = s'(c) = 75 \frac{\text{miles}}{\text{hour}} \quad \leftarrow \text{speeding ticket}$$

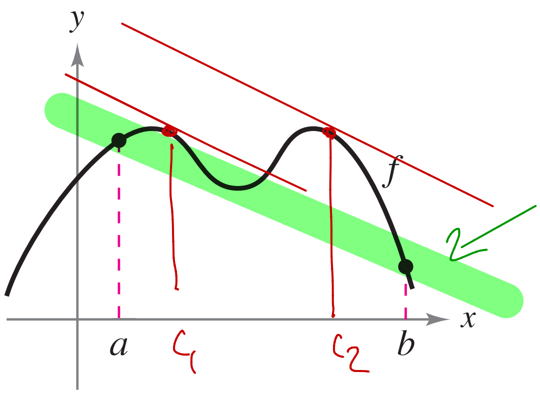
A useful alternative form of the Mean Value Theorem is as follows: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that

$$f(b) = f(a) + (b - a)f'(c).$$

Alternative form of Mean Value Theorem

NOTE When doing the exercises for this section, keep in mind that polynomial functions, rational functions, and trigonometric functions are differentiable at all points in their domains.

Ex.5 Sketch the secant line to the graph through the points $(a, f(a))$ and $(b, f(b))$. Then sketch an tangent lines to the graph for each value of c guaranteed by the Mean Value Theorem.

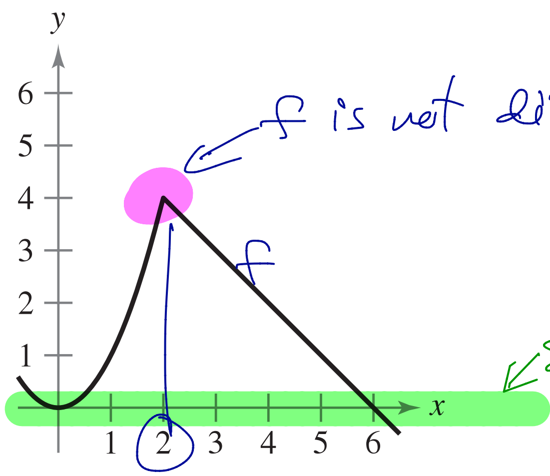


secant line

$$m_{sec} = \frac{f(b) - f(a)}{b - a}$$

$$f'(c_1) = f'(c_2) = m_{sec}$$

Ex.6 Explain why the Mean Value Theorem does not apply to the function f on the interval $[0,6]$.



f is not differentiable on $[0,6]$

secant line through $(0,0)$ & $(6,0)$

$m_{sec} = 0$